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A Study on the Geometry of Einstein Universe

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Abstract :

In this paper, we shall discuss different geometries of the universe considering the line element given by Einstein.

Introduction :

The line element given by Einstein is

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right) dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + dt^2 \tag{1}$$

The equation (1) can be transformed into different forms according to the transformation of co-ordinates.

Transformation of Co-ordinates :

Taking $r = \dfrac{\rho}{1 + \dfrac{\rho^2}{4R^2}}$ as the transformation

We shall get $dr = \dfrac{1 - \dfrac{\rho^2}{4R^2}}{1 + \dfrac{\rho^2}{4R^2}}\, d\rho$

so that equation (1) will transform to

$$ds^2 = -\left(1 + \frac{\rho^2}{4R^2}\right)^{-2} (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta \, d\phi^2) + dt^2 \tag{2}$$

Which can also be transformed into

$$ds^2 = -\left(1 + \frac{\rho^2}{4R^2}\right)^{-2} (dx^2 + dy^2 + dz^2) + dt^2 \tag{3}$$

Now considering the transformations

$$z_1 = R - \left(1 - \frac{r^2}{R^2}\right)$$

$$z_2 = r \sin\theta \cos\phi$$

$$z_3 = r \sin\theta \sin\phi$$

$$z_4 = r \cos\theta$$

We find that equation (1) takes the form

$$ds^2 = -(dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2) + dt^2 \tag{4}$$

where $z_1^2 + z_2^2 + z_3^2 + z_4^2 = R^2$

Equation (4) suggests that the physical space of Einstein universe may be embedded in Euclidean space of higher dimensions.

Spherical Space :

By the transformation $r = R \sin\beta$, equation (1) becomes

$$ds^2 = -R^2 [d\beta^2 + \sin^2\beta \, (d\theta^2 + \sin^2\theta \, d\phi^2)] + dt^2 \tag{5}$$

Here $0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi$

The equation (5) remains unchanged for $\beta = 0$ and $\beta = \pi$, θ and ϕ being arbitrary. This means that corresponding to an event at $\beta = 0$, there exists a mirror image at $\beta = \pi$. In this sense the Einstein universe is taken to be spherical.

The total proper spatial volume of Einstein universe is

$$\int_{\beta=0}^{\beta=\pi} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{2\pi} (Rd\beta)\,(R\sin\beta\, d\theta)\,(R\sin\theta\sin\beta\, d\phi)$$

$$= 4\pi R^3 \int_0^{\pi} \frac{1}{2}(1 - \cos^2\beta)d\beta = 2\pi R^3 \left(\beta - \frac{1}{2}\sin^2\beta\right)_0^{\pi}$$

$$= 2\pi^2 R^3$$

Elliptical Space :

Alternatively, we can take the two events at $\beta = 0$ and $\beta = \pi$, one and the same. In this sense Einstein universe is taken to be elliptical. The total volume of this universe is

$$\int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{dr.\, rd\theta.\, r\sin\theta\, d\phi}{\sqrt{1 - \frac{r^2}{R^2}}}$$

$$= 4\pi \int_0^{R} \frac{r^2 dr}{\sqrt{1 - \frac{r^2}{R^2}}}$$

$$= 4\pi \int_0^{\pi/2} \frac{R^2\sin^2\eta\, R\cos\eta\, d\eta}{\cos\eta}, \text{ where } \frac{r}{R} = \sin\eta$$

$$= 2\pi R^3 \int_0^{\pi/2} (1 - \cos^2\eta)\, d\eta$$

$$= 2\pi R^3 \left(\eta - \frac{1}{2}\sin^2\eta\right)_0^{\pi/2}$$

$$= \pi^2 R^3$$

Here we see that the total proper volume and the total proper distance for elliptical space are just half of the corresponding quantities for spherical space.

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